

# Decay widths in the massive Schwinger model

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## Abstract

By a closer inspection of the massive Schwinger model within mass perturbation theory we find that, in addition to the  $n$ -boson bound states, a further type of hybrid bound states has to be included into the model. Further we explicitly compute the decay widths of the three-boson bound state and of the lightest hybrid bound state.

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# 1 Introduction

The massive Schwinger model is two-dimensional QED with one massive fermion. In this model there are instanton-like gauge field configurations present, and, therefore, a  $\theta$  vacuum has to be introduced as a new, physical vacuum ([1, 2]). Further, confinement is realized in this model in the sense that there are no fermions in the physical spectrum ([3, 4]). The fermions form charge neutral bosons, and only the latter ones exist as physical particles. The fundamental particle of the theory is a massive, interacting boson with mass  $\mu = M_1$  (Schwinger boson). In addition, there exist  $n$ -boson bound states. The two-boson bound state is stable (mass  $M_2$ ), whereas the higher bound states may decay into  $M_1$  and  $M_2$  particles ([2]). All these features have been discussed in [5] within mass perturbation theory ([2], [6] – [8]), which uses the exactly soluble massless Schwinger model ([3], [9] – [14]) as a starting point.

Here we will find that another type of unstable bound states has to be included into the theory, namely hybrid bound states composed of  $M_1$  and  $M_2$  particles. In addition, we will compute the decay widths of the  $M_3$  bound state and of the lightest hybrid bound state (which consists of one  $M_1$  and one  $M_2$  and has mass  $M_{1,1}$ ).

## 2 Bound states

For later convenience we define the functions

$$E_{\pm}(x) = e^{\pm 4\pi D_{\mu}(x)} - 1 \quad (1)$$

and their Fourier transforms  $\tilde{E}_{\pm}(p)$ , where  $D_{\mu}(x)$  is the massive scalar propagator. As was discussed in [5], all the  $n$ -boson bound state masses  $M_n$  may be inferred from the two-point function ( $P = \bar{\Psi}\gamma_5\Psi$ ,  $S = \bar{\Psi}\Psi$ ,  $S_{\pm} = \bar{\Psi}\frac{1}{2}(1 \pm \gamma_5)\Psi$ )

$$\Pi(x) := \delta(x) + g\langle P(x)P(0) \rangle \quad (2)$$

(or  $g\langle S(x)S(0) \rangle$  for even bound states) where  $g = m\Sigma + o(m^2)$  is the coupling constant of the mass perturbation theory for vanishing vacuum angle  $\theta = 0$  (the general  $\theta$  case we discuss in a moment);  $\Sigma$  is the fermion condensate of the massless model.  $\Pi(x)$  is related to the bosonic  $n$ -point functions of the theory via the Dyson-Schwinger equations [5]. In momentum space  $\tilde{\Pi}(p)$  may be resummed,

$$\tilde{\Pi}(p) = \frac{1}{1 - g\langle \widetilde{PP} \rangle_{\text{n.f.}}(p)} \quad (3)$$

where n.f. means non-factorizable and denotes all Feynman graphs that may not be factorized in momentum space. In lowest order  $\langle \widetilde{PP} \rangle_{\text{n.f.}}$  is

$$\langle \widetilde{PP} \rangle_{\text{n.f.}}(p) = \frac{1}{2}(\tilde{E}_+(p) - \tilde{E}_-(p)) \quad (4)$$

(and with a + for  $\langle \widetilde{SS} \rangle_{\text{n.f.}}$ ). Expanding the exponential one finds  $1 - g \sum_{n=1}^{\infty} \frac{(4\pi)^n}{n!} \widetilde{D}_\mu^n(p)$  in the denominator of (3) (more precisely, the odd powers for  $\langle \widetilde{PP} \rangle_{\text{n.f.}}$ , the even powers for  $\langle \widetilde{SS} \rangle_{\text{n.f.}}$ ). At  $p^2 = (n\mu)^2$ ,  $\widetilde{D}_\mu^n(p)$  is singular, therefore there are mass poles  $p^2 = M_n^2$  slightly below the  $n$ -boson thresholds. Further  $\widetilde{D}_\mu^m(p)$  have imaginary parts at  $p^2 = M_n^2$  for  $m < n$ , therefore decays into  $mM_1$  are possible (more precisely, for the parity conserving case  $\theta = 0$ , only odd  $\rightarrow$  odd or even  $\rightarrow$  even decays are possible).

Up to now we did not mention the  $M_2$  particle, although decays into some  $M_2$  are perfectly possible. So where is it? The boson bound states are found by a resummation, so a further resummation is a reasonable idea. Let us look at the  $M_1 + M_2$  final state for definiteness. Within  $g\langle \widetilde{PP} \rangle_{\text{n.f.}}(p)$  we may find the following term

$$H(p) := \int \frac{d^2q}{(2\pi)^2} g\langle \widetilde{PP} \rangle_{\text{n.f.}}(q) \widetilde{\Pi}(q) g\langle \widetilde{PP} \rangle_{\text{n.f.}}(q) 4\pi \widetilde{D}_\mu(p - q). \quad (5)$$

It is simply a loop where we have selected one boson to run along the first line, whereas the  $g\langle \widetilde{PP} \rangle_{\text{n.f.}}(q)$  and  $\widetilde{\Pi}(q)$  run along the other line.  $H(p)$  is a loop, therefore it is non-factorizable. The additional  $(g\langle \widetilde{PP} \rangle_{\text{n.f.}}(q))^2$  factor is necessary, because  $\widetilde{\Pi}(q)$  starts at zeroth order ( $\widetilde{\Pi}(q) = 1 + g\langle \widetilde{PP} \rangle_{\text{n.f.}}(q) + \dots$ ), and without this factor we would include some diagrams into  $H(p)$  that were already used for the  $n$ -boson bound-state formation (double counting).

The claim is that  $H(p)$  has a threshold singularity precisely at  $p^2 = (M_1 + M_2)^2$ , and therefore an imaginary part for  $p^2 > (M_1 + M_2)^2$ . But this is easy to see. At  $q^2 = M_2^2$   $\widetilde{\Pi}(q)$  has the  $M_2$  one-particle singularity and  $g\langle \widetilde{PP} \rangle_{\text{n.f.}}(M_2) \equiv 1$ . Therefore, near  $p^2 = (M_1 + M_2)^2$ ,  $H(p)$  is just the  $M_1, M_2$ - two-boson loop (up to a normalization constant).

Observe that this line of reasoning is not true for higher bound states,  $p^2 \simeq (M_n + M_1)^2$ ,  $n > 2$ .  $\widetilde{\Pi}(M_n)$  contains imaginary parts and is not singular for  $n > 2$  (because the  $M_n$  are unstable), and therefore  $H(p)$  has no thresholds at higher  $p^2$ .

A further consequence is that  $H(p)$  gives rise to a further mass pole slightly below  $p^2 = (M_1 + M_2)^2$  in (3).

These considerations may be generalized, and we find  $n_1 M_1 + n_2 M_2$  particle-production thresholds at  $p^2 = (n_1 M_1 + n_2 M_2)^2$  and (unstable)  $n_1 M_1 + n_2 M_2$ -bound states slightly below.

After all, this is not so surprising. The  $M_2$  are stable particles and interacting via an attractive force. In two dimensions this *must* give rise to a bound state formation. (Similar conclusions may be drawn from unitarity when  $M_2$ -scattering is considered, [15].)

Before starting the actual computations, we should generalize to arbitrary  $\theta \neq 0$ . There the coupling constant is complex,  $g \rightarrow g_\theta, g_\theta^*$ , and, because of parity violation, the Feynman rules acquire a matrix structure (the propagators are  $2 \times 2$  matrices, the vertices tensors, etc.). The exact propagator may be inverted, analogously to (3), and leads to (see [5])

$$\frac{\mathcal{M}_{ij}}{1 - \alpha - \alpha^* + \alpha\alpha^* - \beta\beta^*} \quad (6)$$

$$\alpha(p) = g_\theta \langle \widetilde{S_+ S_+} \rangle_{\text{n.f.}}(p) \quad , \quad \beta(p) = g_\theta \langle \widetilde{S_+ S_-} \rangle_{\text{n.f.}}(p) \quad (7)$$

and  $\mathcal{M}_{ij}$  ( $i, j = +, -$ ) gives the  $\langle \widetilde{S}_i \widetilde{S}_j \rangle$  component of the propagator. For our considerations only the denominator in (6) is important. In leading order

$$\alpha(p) = g_\theta \widetilde{E}_+(p) \quad , \quad \beta(p) = g_\theta \widetilde{E}_-(p) \quad , \quad g_\theta = \frac{m\Sigma}{2} e^{i\theta} \quad (8)$$

and the denominator reads

$$1 - m\Sigma \cos \theta \widetilde{E}_+(p) + \frac{m^2 \Sigma^2}{4} (\widetilde{E}_+^2(p) - \widetilde{E}_-^2(p)). \quad (9)$$

Inserting the  $n$ -boson functions ( $d_n(p) := \frac{(4\pi)^n}{n!} \widetilde{D}_\mu^n(p)$ ) results in

$$1 - m\Sigma \cos \theta (d_1 + d_2 + \dots) + m^2 \Sigma^2 (d_1(d_2 + d_4 + \dots) + d_3(d_2 + d_4 + \dots) + \dots) \quad (10)$$

Now suppose we are e.g. at the  $M_3$  bound state mass. Then the real part of (10) vanishes by definition and  $m\Sigma \cos \theta d_3(M_3) = 1 + o(m)$ , and we get

$$-im\Sigma \cos \theta \text{Im } d_2(M_3) + im^2 \Sigma^2 d_3(M_3) \text{Im } d_2(M_3) = -im\Sigma (\cos \theta - \frac{1}{\cos \theta}) \text{Im } d_2(M_3). \quad (11)$$

This computation may be generalized easily, and we find that each parity allowed decay acquires a  $\cos \theta$ , whereas a parity forbidden decay acquires a  $(\cos \theta - \frac{1}{\cos \theta})$  factor.

To include the decays into  $M_2$  we have to perform a further resummation analogous to above, however, the resummed contributions enter into the functions  $\alpha, \beta$  in a way that is perfectly consistent with our parity considerations (a  $n_1 M_1 + n_2 M_2$ -state has parity  $P = (-1)^{n_1}$ ).

### 3 Bound state masses

We are now prepared for explicit computations, but before computing decay widths we need the masses and residues of the propagator at the various mass poles. The masses  $M_1, M_2, M_3$  have already been computed ([5]; there is, however, a numerical error in the  $M_2$  mass formula in [5]),

$$M_1^2 \equiv \mu^2 = \mu_0^2 + \Delta_1 + o(m^2) \quad , \quad \Delta_1 = 4\pi m\Sigma \cos \theta \quad (12)$$

$$M_2^2 = 4\mu^2 - \Delta_2 \quad , \quad \Delta_2 = \frac{4\pi^4 m^2 \Sigma^2 \cos^2 \theta}{\mu^2} \quad (13)$$

$$M_3^2 = 9\mu^2 - \Delta_3 \quad , \quad \Delta_3 \simeq 6.993\mu^2 \exp(-0.263 \frac{\mu^2}{m\Sigma \cos \theta}) \quad (14)$$

and the three-boson binding energy is smaller than polynomial in the coupling constant  $m$  (or  $g$ ).

In leading order the  $n$ -th mass pole is the zero of the function

$$f_n(p^2) = 1 - m\Sigma \cos \theta d_n(p^2), \quad (15)$$

therefore the residue may be inferred from the first Taylor coefficient around  $(p^2 - M_n^2)$ ,

$$f_n(p^2) \simeq c_n(p^2 - M_n^2). \quad (16)$$

The  $c_n$  may be inferred from the computation of the mass poles ([5]) and are related to the binding energies. Explicitly they read

$$c_1 = \frac{1}{4\pi m \Sigma \cos \theta} = \frac{1}{\Delta_1} \quad (17)$$

$$c_2 = \frac{\mu^2}{8\pi^4 (m \Sigma \cos \theta)^2} = \frac{1}{2\Delta_2} \quad (18)$$

$$c_3 = \frac{m \Sigma \cos \theta}{0.263 \mu^2 \Delta_3} \quad (19)$$

The mass  $M_{1,1}$  is the solution of  $1 = (g_\theta + g_\theta^*)H(p)$ , which looks difficult to solve. However, there is an approximation. At threshold  $\bar{\Pi}(q)$  equals the  $M_2$  propagator, so this may be a reasonable approximation provided that the binding energy is sufficiently small,  $\Delta_{1,1} \equiv (M_1 + M_2)^2 - M_{1,1}^2 < \Delta_2$ . In this approximation we have for  $M_{1,1}$

$$\begin{aligned} 1 &= m \Sigma \cos \theta \int \frac{d^2 q}{(2\pi)^2} \frac{8\pi^4 m \Sigma \cos \theta}{\mu^2 (q^2 - M_2^2)} \frac{4\pi}{(p - q)^2 - M_1^2} \\ &= \frac{32\pi^5 m^2 \Sigma^2 \cos^2 \theta}{2\pi \mu^2 \bar{w}(p^2, M_2^2, M_1^2)} \left( \pi + \right. \\ &\quad \left. \arctan \frac{2p^2}{\bar{w}(p^2, M_2^2, M_1^2) - \frac{1}{\bar{w}(p^2, M_2^2, M_1^2)}(p^2 + M_1^2 - M_2^2)(p^2 - M_1^2 + M_2^2)} \right) \end{aligned} \quad (20)$$

$$\bar{w}(x, y, z) := (-x^2 - y^2 - z^2 + 2xy + 2xz + 2yz)^{\frac{1}{2}} \quad (21)$$

where we inserted the residues that may be derived from the Taylor coefficients  $c_1, c_2$  (17,18) ( $\text{Res}_i = \frac{1}{c_i m \Sigma \cos \theta}$ ). The solution is

$$M_{1,1}^2 = (M_1 + M_2)^2 - \Delta_{1,1} \quad , \quad \Delta_{1,1} = \frac{32\pi^{10} (m \Sigma \cos \theta)^4}{\mu^6} \quad (22)$$

which shows that our approximation is justified for sufficiently small  $m$ .

$M_{1,1}$  was computed in a way analogous to  $M_2$  (see [5]), therefore it leads to an analogous Taylor coefficient

$$c_{1,1} = \frac{1}{2\Delta_{1,1}} = \frac{\mu^6}{64\pi^{10} (m \Sigma \cos \theta)^4}. \quad (23)$$

## 4 Decay width computation

The decay widths may be inferred in a simple way from the imaginary parts of the propagator. Generally

$$G(p) \sim \frac{\text{const.}}{p^2 - M^2 - i\Gamma M} \quad (24)$$

and  $\Gamma$  is the decay width. In our case the poles have their Taylor coefficients,

$$\tilde{\Pi}(p) \sim \frac{\text{const.}}{c_i(p^2 - M_i^2) - i\text{Im}(\dots)} \sim \frac{\text{const.}'}{p^2 - M_i^2 - i\frac{\text{Im}(\dots)}{c_i}} \quad (25)$$

and therefore

$$\Gamma_i \sim \frac{\text{Im}(\dots)}{c_i M_i}. \quad (26)$$

Before performing the explicit computations let us add a short remark. The  $c_i$  are related to the binding energies,  $c_i \sim \frac{1}{\Delta_i}$ . Therefore, all the decay widths are restricted by the binding energies,  $\Gamma_i \sim \Delta_i$ . But this is a very reasonable result. The denominator of the propagator (25) has zero real part at  $M_i^2$  and infinite real part at the real particle production threshold. Suppose  $\tilde{\Pi}(p)$  contributes to a scattering process (to be discussed in detail in a further publication, [15]). It will give rise to a local maximum (resonance) at  $p^2 = M_i^2$ , and to a local minimum at the production threshold  $p^2 = M_i^2 + \Delta_i$ . Therefore the resonance width (decay width) *must* be bounded by  $\Delta_i$ .

Now let us perform the explicit calculations. At  $M_{1,1}^2$  the propagator is

$$\tilde{\Pi}(p) \sim \frac{1}{c_{1,1}(p^2 - M_{1,1}^2) - im\Sigma(\cos\theta - \frac{1}{\cos\theta})\text{Im } d_2(p)} \quad (27)$$

$$\text{Im } d_2(p) = \frac{8\pi^2}{2w(p^2, M_1^2, M_1^2)}$$

$$w(x, y, z) = (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{\frac{1}{2}} \quad (28)$$

leading to the decay width ( $M_1 \equiv \mu$ )

$$\Gamma_{M_{1,1}} = \frac{2^8 \pi^{12} (m\Sigma \cos\theta)^5}{9\sqrt{5}\mu^9} \left( \frac{1}{\cos^2\theta} - 1 \right) \simeq 21340\mu \left( \frac{m \cos\theta}{\mu} \right)^5 \left( \frac{1}{\cos^2\theta} - 1 \right) \quad (29)$$

( $\Sigma = \frac{e\gamma\mu}{2\pi} = 0.283\mu$ ) for the decay  $M_{1,1} \rightarrow 2M_1$ . This decay is parity forbidden, and therefore  $M_{1,1}$  is stable for  $\theta = 0$ .

For the  $M_3$  decay there exist two channels,  $M_3 \rightarrow M_2 + M_1, M_3 \rightarrow 2M_1$ ,

$$\tilde{\Pi}(p) \sim \frac{1}{c_3(p^2 - M_3^2) - im\Sigma(\cos\theta - \frac{1}{\cos\theta})\frac{4\pi^2}{w(p^2, M_1^2, M_1^2)} - i(m\Sigma \cos\theta)^2 \frac{16\pi^5}{\mu^2 w(p^2, M_2^2, M_1^2)}} \quad (30)$$

leading to the partial decay widths

$$\Gamma_{M_3 \rightarrow 2M_1} = 0.263 \frac{4\pi^2 \Delta_3}{9\sqrt{5}\mu} \left( \frac{1}{\cos^2\theta} - 1 \right) \simeq 3.608\mu \left( \frac{1}{\cos^2\theta} - 1 \right) \exp(-0.929 \frac{\mu}{m \cos\theta}) \quad (31)$$

$$\Gamma_{M_3 \rightarrow M_2 + M_1} = 0.263 \frac{4\pi^3 \Delta_3}{3\sqrt{3}\mu} \simeq 43.9\mu \exp(-0.929 \frac{\mu}{m \cos \theta}) \quad (32)$$

and to the ratio

$$\frac{\Gamma_{M_3 \rightarrow 2M_1}}{\Gamma_{M_3 \rightarrow M_2 + M_1}} = \frac{\frac{1}{\cos^2 \theta} - 1}{\sqrt{15}\pi}. \quad (33)$$

The latter is independent of the approximations that were used in the computation of  $M_3$  and  $c_3$ . Observe that  $\Gamma_{M_3 \rightarrow M_2 + M_1}$  is larger than  $\Gamma_{M_3 \rightarrow 2M_1}$ , although  $M_1 + M_2 \sim M_3$ . This is so because the phase space "volume" does not rise with increasing momentum in  $d = 1 + 1$ .

Remark: there seems to be a cheating concerning the sign of  $\Gamma_{M_3 \rightarrow 2M_1}$  (see (30), (31)). This is a remnant of the Euclidean conventions that are implicit in our computations (see e.g. [5]). There the conventions are such that  $\theta$  is imaginary and, consequently,  $\cos \theta - \frac{1}{\cos \theta} \geq 0$ . In a really Minkowskian computation, roughly speaking, the roles of  $E_+$  and  $E_-$  in (6) are exchanged, leading to a relative sign between odd and even states. The final results (29), (31) and (32) are expressed for Minkowski space and for real  $\theta$  ( $\frac{1}{\cos^2 \theta} - 1 \geq 0$ ), which explains the sign.

## 5 Summary

By a closer inspection of the massive Schwinger model we have found that its spectrum is richer than expected earlier. In addition to the  $n$ -boson bound states there exist hybrid bound states that are composed of fundamental bosons and stable two-boson bound states. A posteriori their existence is not too surprising and may be traced back to the fact that particles that attract each other form at least one bound state in  $d = 2$ ; or it may be understood by some unitarity arguments. For the special case of vanishing vacuum angle,  $\theta = 0$ , the lowest of these hybrid bound states is even stable and must be added to the physical particles of the theory.

Further we computed the decay widths of some unstable bound states and found that our results are consistent with an interpretation of the bound states as resonances. Even more insight into these features would be possible by a discussion of scattering, which will be done in a forthcoming publication ([15]).

Of course, it would be interesting to compare our results to other approaches, like e.g. lattice calculations.

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## References

- [1] S. Coleman, R. Jackiw, L. Susskind, *Ann. Phys.* *93* (1975) 267
- [2] S. Coleman, *Ann. Phys.* *101* (1976) 239
- [3] E. Abdalla, M. Abdalla, K. D. Rothe, "2 dimensional Quantum Field Theory", World Scientific, Singapore, 1991
- [4] D. J. Gross, I. R. Klebanov, A. V. Matytsin, A. V. Smilga, *Nucl. Phys. B* *461* (1996) 109, hep-th 9511104
- [5] C. Adam, *Phys. Lett. B* *382* (1996) 111; hep-th 9602175
- [6] J. Fröhlich, E. Seiler, *Helv. Phys. Acta* *49* (1976) 889
- [7] C. Adam, *Phys. Lett. B* *363* (1995) 79; hep-ph 9507279
- [8] C. Adam, *Phys. Lett. B* *382* (1996) 383; hep-ph 9507331
- [9] J. Schwinger, *Phys. Rev.* *128* (1962) 2425
- [10] J. Lowenstein, J. Swieca, *Ann. Phys.* *68* (1971) 172
- [11] C. Jayewardena, *Helv. Phys. Acta* *61* (1988) 636
- [12] I. Sachs, A. Wipf, *Helv. Phys. Acta* *65* (1992) 653
- [13] C. Adam, R. A. Bertlmann, P. Hofer, *Riv. Nuovo Cim.* *16*, No 8 (1993)
- [14] C. Adam, *Z. Phys. C* *63* (1994) 169
- [15] C. Adam, "Scattering processes in the massive Schwinger model", to be published